§ Maxima and Minima: (Second order derivative Test)

Conditions for Maximum value:

A function $y = f(x)$ has maximum value $f(a)$ at the point $x = a$ if

(i) $\frac{dy}{dx} = 0$ at $x = a$ and

(ii) $\frac{d^2y}{dx^2}$ is negative ie $\frac{d^2y}{dx^2} < 0$, at $x = a$

Conditions for Minimum value: (Second order derivative Test)

A function $y = f(x)$ has minimum value $f(a)$ at the point $x = a$ if -

(i) $\frac{dy}{dx} = 0$ at $x = a$ and

(ii) $\frac{d^2y}{dx^2} > 0$ ie $\frac{d^2y}{dx^2}$ is positive at $x = a$

Working Rule for finding The Relative Maxima and Minima:

The method of finding maximum and the minimum value of a function $y = f(x)$ can be summarized in the following steps:

Step-1: Find $f'(x)$ and solve the equation $f'(x) = 0$, for x. Let $x = a, b, c, \ldots$ be its roots.

Step 2: Test the values of x found in step 1 for maxima and minima

i) if $\frac{d^2y}{dx^2} < 0$, for $x = a$ Then The function has maximum value at $x = a$ and $\text{Max } f(x) = f(a)$

ii) if $\frac{d^2y}{dx^2} > 0$, for $x = a$, Then The function has minimum value at the point $x = a$ and $\text{Min } f(x) = f(a)$

Similarly we Test the other values b, c -- found in step 1

Step 3: if $\frac{d^2y}{dx^2} = 0$, for a particular value of $x = a$ (say), then $x = a$ gives a point of inflexion and therefore the function gives neither maximum nor minimum value.

Minimization of cost: We can use the technique of maxima and minima to obtain minimum total cost or average variable cost. Let the total cost function be given by $c = f(q)$, where c denotes total cost and q denotes output.

Condition for cost minimization:

a) **To minimize Total cost**: The following conditions must be fulfilled to obtain minimum total cost (TC)

i) $\frac{dc}{dq} = 0$. (ii) $\frac{d^2c}{dq^2} > 0$

b) **To minimize Average cost**: The following conditions must be fulfilled to obtain minimum average cost (AC)

i) $\frac{d}{dq}(AC) = 0$ ii) $\frac{d^2}{dq^2}(AC) > 0$

c) **To minimize Average variable cost**: To obtain minimum average variable cost (AVC), the following conditions must be satisfied.

i) $\frac{d}{dq}(AVC) = 0$ ii) $\frac{d^2}{dq^2}(AVC) > 0$

d) **To minimize Marginal cost (MC)**: To obtain minimum marginal cost (MC), the following conditions must be satisfied -

(i) $\frac{d}{dq}(MC) = 0$ (ii) $\frac{d^2}{dq^2}(MC) > 0$

Ex: The cost c per mile of an electric cable is given by $c = \frac{120}{x} + 600x$; where x is the cross section in square inches. Find the cross section for which the cost is least and find the least cost per mile.

Soln: Given, $c = \frac{120}{x} + 600x$

$$\therefore \frac{dc}{dx} = \frac{d}{dx}\left(\frac{120}{x} + 600x\right)$$

$$= 120 \cdot \frac{d}{dx}\left(\frac{1}{x}\right) + 600\frac{d}{dx}(x)$$

$$= 120 \cdot \left(-\frac{1}{x^2}\right) + 600 \cdot 1$$

$$= -\frac{120}{x^2} + 600$$

$$\frac{d^2c}{dx^2} = \frac{d}{dx}\left(\frac{dc}{dx}\right) = \frac{d}{dx}\left(-\frac{120}{x^2} + 600\right) = -120\frac{d}{dx}\left(\frac{1}{x^2}\right) + \frac{d}{dx}(600)$$

$$= -120\left(-\frac{2}{x^3}\right) + 0$$

$$= \frac{240}{x^3}$$

For extreme values,

$$\frac{dc}{dx} = 0 \Rightarrow -\frac{120}{x^2} + 600 = 0$$

$$\Rightarrow \frac{120}{x^2} = 600 \Rightarrow x^2 = \frac{\cancel{120}}{\cancel{600}} = \frac{1}{5}$$

$$\therefore x = \pm\frac{\sqrt{1}}{5}. \quad \because x \neq -ve \quad \therefore x = \frac{1}{\sqrt{5}}$$

at $x = \frac{1}{\sqrt{5}}$, $\frac{d^2c}{dx^2} = \frac{240}{(\sqrt{5})^3} = \frac{\cancel{240}^{48}}{8\sqrt{5}} = \frac{48}{\sqrt{5}} > 0$

$\therefore x = \frac{1}{\sqrt{5}}$ gives the least (minimum) cost and the least (minimum) cost is given by -

$$C = \frac{120}{\frac{1}{\sqrt{5}}} + 600 \times \frac{1}{\sqrt{5}} = 120\sqrt{5} + \frac{120 \times 5}{\sqrt{5}} = 120\sqrt{5} + 120\sqrt{5} = 240\sqrt{5}$$

Ex: A manufacturer produces x tons of steel per week at a total cost of Rs $x^2 - 10x + 100$. Find the output level for which the total cost attains minimum and find the total cost.

Soln: Here Total cost (TC) $= x^2 - 10x + 100$

$$\therefore \frac{d}{dx}(TC) = \frac{d}{dx}(x^2 - 10x + 100)$$
$$= \frac{d}{dx}(x^2) - 10 \cdot \frac{d}{dx}(x) + \frac{d}{dx}(100)$$
$$= 2x - 10 \cdot 1 + 0$$
$$= 2x - 10$$

$$\frac{d^2}{dx^2}(TC) = \frac{d}{dx}\left[\frac{d(TC)}{dx}\right] = \frac{d}{dx}(2x - 10)$$
$$= 2 \cdot \frac{d}{dx}(x) - \frac{d}{dx}(10)$$
$$= 2 \cdot 1 - 0$$
$$= 2$$

Now, for minimization,

$$\frac{d(TC)}{dx} = 0 \Rightarrow 2x - 10 = 0 \Rightarrow 2x = 10 \Rightarrow x = 5$$

$\therefore$ at $x = 5$, $\frac{d^2(TC)}{dx^2} = 2 > 0$

$\therefore$ output $x = 5$ gives the minimum total cost and the total cost is $5^2 - 10 \cdot 5 + 100 = 75$

$\therefore$ Required Production = 5 tons, The minimum total cost = 75

Maximisation of Revenue: We know that revenue indicates sales receipts. The total revenue of a firm is $R = pq$, where p denotes price and q the quantity

Condition for Revenue Maximization: The following conditions are to be obtained for maximimum revenue

(i) $\frac{dR}{dq} = 0$ (ii) $\frac{d^2R}{dq^2} < 0$

To obtain maximum average revenue (AR), The following conditions are to be fulfilled

(i) $\frac{d}{dq}(AR) = 0$ (ii) $\frac{d^2(AR)}{dq^2} < 0$

Ex (27) If $p = \frac{121}{q+4} - 1$; find the output level at which total revenue is maximum. Also find the maximum revenue

Soln:- As $p = \frac{121}{q+4} - 1$

$\therefore$ Total revenue $= R = pq = \left[\frac{121}{q+4} - 1\right] \cdot q = \frac{121 \cdot q}{q+4} - q$

Now, $\frac{dR}{dq} = \frac{d}{dq}\left[\frac{121q}{q+4} - q\right] = \frac{d}{dq}(121q) - \frac{d}{dq}(q)$

$= 121 \cdot \frac{d}{dq}\left(\frac{q}{q+4}\right) - 1$

$= 121 \cdot \frac{(q+4)\frac{d}{dq}(q) - q\frac{d}{dq}(q+4)}{(q+4)^2} - 1$

$= 121 \cdot \frac{q+4 - q \cdot 1}{(q+4)^2} - 1$

$= \frac{121 \times 4}{(q+4)^2} - 1$

$= \frac{484}{(q+4)^2} - 1$

Again, $\frac{d^2R}{dq^2} = \frac{d}{dq}\left(\frac{dR}{dq}\right) = \frac{d}{dq}\left[\frac{484}{(q+4)^2} - 1\right]$

$= 484 \frac{d}{dq}\left[\frac{1}{(q+4)^2}\right] - \frac{d}{dq}(1)$

$= 484\left[\frac{-2}{(q+4)^3} \cdot 1\right] - 0$

$= -\frac{968}{(q+4)^3}$

For extreme values

$\frac{dR}{dq} = 0 \Rightarrow \frac{484}{(q+4)^2} - 1 = 0 \Rightarrow \frac{484}{(q+4)^2} = 1$

$\Rightarrow (q+4)^2 = 484$

$\Rightarrow (q+4)^2 = (22)^2$

$\Rightarrow q+4 = 22$

$\Rightarrow q = 22-4$

$= 18$

Now, at $q = 18$, $\frac{d^2R}{dq^2} = -\frac{968}{(18+4)^3} < 0$

$\therefore$ $q = 18$ gives the maximum revenue.

$\therefore$ Maximum Revenue $= \frac{121 \times 18}{18+4} - 18$

$= \frac{2178}{22} - 18$

$= 99 - 18$

$= 81$

Maximization of Profit: We can use the technique of maxima and minima to obtain level of output at which the profit of a firm is maximum.

We know that, Profit (π) = Total revenue − Total cost = R − C

Conditions for maximum Profit: To obtain maximum profit, the following condition must be satisfied

(i) $\frac{d\pi}{dq} = 0 \Rightarrow \frac{d}{dq}(R-C) = 0 \Rightarrow \frac{d}{dq}(R) - \frac{d}{dq}(C) = 0 \Rightarrow MR = MC$

(ii) $\frac{d^2\pi}{dq^2} < 0 \Rightarrow \frac{d^2R}{dq^2} - \frac{d^2C}{dq^2} < 0 \Rightarrow \frac{d^2R}{dq^2} < \frac{d^2C}{dq^2}$

$\Rightarrow \frac{d^2C}{dq} > \frac{d^2R}{dq^2}$

Ex: A monopolist has the following demand and cost function, $P = 80-3q$, and $C = q^2+8q+6$ respectively. Find the profit maximizing output and the maximum profit.

Soln: Total revenue, $R = pq = (80-3q).q = 80q-3q^2$

Total cost (C) $= q^2+8q+6$

$\therefore$ Profit (π) $= R - C$

$= (80q-3q^2)-(q^2+8q+6)$

$= 80q-3q^2-q^2-8q-6$

$= 72q-4q^2-6$

$\therefore \frac{d\pi}{dq} = \frac{d}{dq}(72q-4q^2-6)$

$= 72.\frac{d}{dq}(q) - 4\frac{d}{dq}(q^2) - \frac{d}{dq}(6)$

$= 72.1 - 4.2q - 0$

$= 72-8q$

Again $\frac{d^2\pi}{dq^2} = \frac{d}{dq}\left(\frac{d\pi}{dq}\right) = \frac{d}{dq}(72-8q) = \frac{d}{dq}(72) - 8.\frac{d(q)}{dq} = 0-8.1 = -8$

For extreme values, $\frac{d\pi}{dq} = 0 \Rightarrow 72-8q = 0 \Rightarrow 8q = 72$
$\Rightarrow q = 9$

$\therefore$ q = 9 gives the maximum Profit if at q = 9, $\frac{d^2\pi}{dq^2} < 0$

Now, at $q = 9$, $\frac{d^2\pi}{dq^2} = -8 < 0$

$\therefore$ q = 9 gives the maximum profit and the maximum profit is $= 72\times9 - 4.9^2 - 6$

$= 648 - 4.81 - 6$

$= 648 - 324 - 6$

$= 648 - 330$

$= 318$