c) The derivative of the sum or difference of two functions.

Let $y = f(x) = f_1(x) + f_2(x)$

$\therefore f(x+\partial x) = f_1(x+\partial x) + f_2(x+\partial x)$

$\therefore \partial y = f(x+\partial x) - f(x)$

$\Rightarrow \dfrac{\partial y}{\partial x} = \dfrac{f(x+\partial x) - f(x)}{\partial x}$

$\Rightarrow \dfrac{\partial y}{\partial x} = \dfrac{f_1(x+\partial x) + f_2(x+\partial x) - [f_1(x) + f_2(x)]}{\partial x}$

$\Rightarrow \lim\limits_{\partial x \to 0} \dfrac{\partial y}{\partial x} = \lim\limits_{\partial x \to 0} \dfrac{f_1(x+\partial x) + f_2(x+\partial x) - f_1(x) - f_2(x)}{\partial x}$

$\Rightarrow \dfrac{dy}{dx} = \lim\limits_{\partial x \to 0} \dfrac{[f_1(x+\partial x) - f_1(x)] + [f_2(x+\partial x) - f_2(x)]}{\partial x}$

$\Rightarrow \dfrac{dy}{dx} = \lim\limits_{\partial x \to 0} \dfrac{f_1(x+\partial x) - f_1(x)}{\partial x} + \lim\limits_{\partial x \to 0} \dfrac{f_2(x+\partial x) - f_2(x)}{\partial x}$

$\Rightarrow \dfrac{d}{dx}[f_1(x) + f_2(x)] = \dfrac{d}{dx}[f_1(x)] + \dfrac{d}{dx}[f_2(x)]$

Similarly, $\dfrac{d}{dx}[f_1(x) - f_2(x)] = \dfrac{d}{dx}[f_1(x)] - \dfrac{d}{dx}[f_2(x)]$

<u>Note</u>: If u and v are the two functions of x, then

$\dfrac{d}{dx}(u \pm v) = \dfrac{d}{dx}(u) \pm \dfrac{d}{dx}(v)$

d) The derivative of the product of two functions

Let $y = f(x) \cdot g(x)$

$y + \partial y = f(x+\partial x) \cdot g(x+\partial x)$

$\therefore y + \partial y - y = f(x+\partial x) \cdot g(x+\partial x) - f(x) \cdot g(x)$

$\Rightarrow \partial y = f(x+\partial x) \cdot g(x+\partial x) - f(x) \cdot g(x)$

$\Rightarrow \dfrac{\partial y}{\partial x} = \dfrac{f(x+\partial x) \cdot g(x+\partial x) - f(x) \cdot g(x)}{\partial x}$

$\Rightarrow \lim\limits_{\partial x \to 0} \dfrac{\partial y}{\partial x} = \lim\limits_{\partial x \to 0} \dfrac{f(x+\partial x) \cdot g(x+\partial x) - f(x) \cdot g(x)}{\partial x}$

$\Rightarrow \dfrac{dy}{dx} = \lim\limits_{\partial x \to 0} \dfrac{f(x+\partial x) \cdot g(x+\partial x) - f(x+\partial x) \cdot g(x) + f(x+\partial x) \cdot g(x) - f(x) \cdot g(x)}{\partial x}$

(20)

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x\to 0} \frac{[f(x+\delta x).g(x+\delta x) - f(x+\delta x).g(x)] + [f(x+\delta x).g(x) - f(x).g(x)]}{\delta x}$$

$$= \lim_{\delta x\to 0} \frac{[f(x+\delta x).g(x+\delta x) - f(x+\delta x).g(x)]}{\delta x} + \lim_{\delta x\to 0} \frac{[f(x+\delta x).g(x) - f(x).g(x)]}{\delta x}$$

$$= \lim_{\delta x\to 0}\left[f(x+\delta x).\frac{g(x+\delta x)-g(x)}{\delta x}\right] + \lim_{\delta x\to 0}\left[g(x).\frac{f(x+\delta x)-f(x)}{\delta x}\right]$$

$$= \lim_{\delta x\to 0}[f(x+\delta x)].\lim_{\delta x\to 0}\frac{g(x+\delta x)-g(x)}{\delta x} + \lim_{\delta x\to 0}[g(x)]\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$= f(x).\frac{d}{dx}[g(x)] + g(x).\frac{d}{dx}[f(x)]$$

$$\therefore \frac{d}{dx}[f(x).g(x)] = f(x).\frac{d}{dx}[g(x)] + g(x).\frac{d}{dx}[f(x)]$$

Rules for differentiation:

1. $\frac{d}{dx}(c) = 0$; where c is a constant
2. $\frac{d}{dx}[c.f(x)] = c.\frac{d}{dx}[f(x)]$; where c is a constant
3. $\frac{d}{dx}[f_1(x) \pm f_2(x) \pm \cdots \pm f_k(x)] = \frac{d}{dx}[f_1(x)] \pm \frac{d}{dx}[f_2(x)] \pm \cdots \pm \frac{d}{dx}[f_k(x)]$
4. $\frac{d}{dx}[f(x).g(x)] = f(x).\frac{d}{dx}[g(x)] + g(x).\frac{d}{dx}[f(x)]$

Formulae for differentiation:

1. $\frac{d}{dx}(x^n) = n.x^{n-1}$
2. $\frac{d}{dx}(\log x) = \frac{1}{x}$
3. $\frac{d}{dx}(e^x) = e^x$
4. $\frac{d}{dx}(a^x) = a^x \log_e a$; where a is any constant other than e. e stands for the value of an infinite series

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$ and its value correct upto five decimal is 2.71828

Ex find $\frac{dy}{dx}$ of $y = 5x^4 - 3x^3 + 4x^2 + 7x - 10$

Soln: $y = 5x^4 - 3x^3 + 4.x^2 + 7x - 10$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[5x^4 - 3x^3 + 4x^2 + 7x - 10\right]$$

$$= \frac{d}{dx}(5x^4) - \frac{d}{dx}(3x^3) + \frac{d}{dx}(4x^2) + \frac{d}{dx}(7x) - \frac{d}{dx}(10)$$

$$= 5.\frac{d}{dx}(x^4) - 3\frac{d}{dx}(x^3) + 4.\frac{d}{dx}(x^2) + 7.\frac{d}{dx}(x) - \frac{d}{dx}(10)$$

$$= 5.4.x^{4-1} - 3.3.x^{3-1} + 4.2.x^{2-1} + 7.1.x^{1-1} - 0 \quad \left[\begin{array}{l} \frac{d}{dx}(x^n) = n.x^{n-1} \\ \frac{d}{dx}(c) = 0 \end{array}\right]$$

$$= 20.x^3 - 9x^2 + 8x + 7 - 0$$

$$= 20x^3 - 9x^2 + 8x + 7$$

Ex: find $\frac{dy}{dx}$ of $y = x^2 \log x$

soln: $y = x^2 \log x$

$$\therefore \frac{dy}{dx} = x^2.\frac{d}{dx}(\log x) + \log x.\frac{d}{dx}(x^2) \quad \left[\text{using product rule of differentiation}\right]$$

$$= x^2.\frac{1}{x} + \log x.2x$$

$$= x + 2x\log x$$

$$= x(1 + 2\log x)$$

Ex: find $\frac{dy}{dx}$ of $y = (x+2)(x+1)^2$

soln: $y = (x+2)(x+1)^2$

$$\Rightarrow y = (x+2)(x^2 + 2x + 1)$$

$$\Rightarrow \frac{dy}{dx} = (x+2).\frac{d}{dx}(x^2 + 2x + 1) + (x^2 + 2x + 1).\frac{d}{dx}(x+2)$$

$$= (x+2)\left[\frac{d}{dx}(x^2) + 2.\frac{d}{dx}(x) + \frac{d}{dx}(1)\right] + (x^2 + 2x + 1)\left[\frac{d}{dx}(x) + \frac{d}{dx}(2)\right]$$

$$= (x+2)(2x + 2 + 0) + (x^2 + 2x + 1).(1 + 0)$$

$$= (x+2)(2x+2) + (x^2 + 2x + 1).1$$

$$= x(2x+2) + 2(2x+2) + x^2 + 2x + 1$$

$$= 2x^2 + 2x + 4x + 4 + x^2 + 2x + 1$$

$$= 3x^2 + 8x + 5$$

Ex Find $\frac{dy}{dx}$ of $y = 2x^2 - 3\log x + 6.e^{2x} + 35.\frac{1}{x}$

Soln: $y = 2x^2 - 3\log x + 6.e^{2x} + 35.\frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[2x^2 - 3\log x + 6.e^{2x} + 35.\frac{1}{x}\right]$

$= 2\frac{d}{dx}(x^2) - 3\frac{d}{dx}(\log x) + 6.\frac{d}{dx}(e^{2x}) + 35\frac{d}{dx}\left(\frac{1}{x}\right)$

$= 2.2x - 3.\frac{1}{x} + 6.e^{2x}.\frac{d}{dx}(2x) + 35.\left(-\frac{1}{x^2}\right)$

$= 4x - \frac{3}{x} + 6.e^{2x}.2 - \frac{35}{x^2}$

$= 4x - \frac{3}{x} + 12.e^{2x} - \frac{35}{x^2}$ //

$\frac{d}{dx}(e^{2x})$
$= e^{2x}.\frac{d}{dx}(2x)$
$= e^{2x}.2\frac{d}{dx}(x)$
$= e^{2x}.2.1$
$= 2.e^{2x}$

$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1})$
$= (-1)x^{-1-1}$
$= -1.x^{-2}$
$= -1\frac{1}{x^2}$
$= -\frac{1}{x^2}$

Ex find $\frac{dy}{dx}$ of $y = x^2.e$

Soln: $y = x^2 e$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^2 e)$

$= e\frac{d}{dx}(x^2)$ [$\because$ e is a constant]

$= e.2x$

$= 2ex$ //

Ex :- find $\frac{dy}{dx}$ of $y = x^n.e^x$

Soln: $y = x^n.e^x$

$\Rightarrow \frac{dy}{dx} = x^n.\frac{d}{dx}(e^x) + e^x.\frac{d}{dx}(x^n)$

$= x^n.e^x + e^x.n.x^{n-1}$ [$\because x^n = x.x^{n-1}$]

$= e^x.x^{n-1}(x+n)$ //

Logarithmic differentiation: If the index of a function f(x) is also a function of x or if the function is the product of two functions then first of all, we are to take the Logarithm of both sides and then proceed to find the derivative.

note: $\frac{d}{dx}(\log y) = \frac{1}{y}.\frac{dy}{dx}$

Ex: Find $\frac{dy}{dx}$ of $y = x^x$

Soln: $y = x^x$

$\Rightarrow \log y = \log(x^x)$

$\Rightarrow \log y = x \cdot \log x$

$\Rightarrow \frac{d}{dx}(\log y) = \frac{d}{dx} x \log x$

$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}(x)$

$\Rightarrow \frac{1}{y} \frac{dy}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1$

$\Rightarrow \frac{1}{y} \frac{dy}{dx} = 1 + \log x$

$\Rightarrow \frac{dy}{dx} = y(1 + \log x)$

$= x^x (1 + \log x)$ //

Ex: Find $\frac{dy}{dx}$ of $y = a^x b^x$

Soln: $y = a^x b^x$

$\Rightarrow \log y = \log(a^x . b^x)$

$\Rightarrow \log y = \log a^x + \log b^x$

$\Rightarrow \log y = x \log a + x \log b$

$\Rightarrow \frac{d}{dx}(\log y) = \frac{d}{dx}(x \log a + x \log b)$

$\Rightarrow \frac{d}{dx}(\log y) = \frac{d}{dx}(x \log a) + \frac{d}{dx}(x \log b)$

$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \log a \frac{d}{dx}(x) + \log b \cdot \frac{d}{dx}(x)$

$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \log a + \log b$

$\Rightarrow \frac{dy}{dx} = y(\log a + \log b)$

$= a^x b^x (\log a + \log b)$ //

Ex: Find $\frac{dy}{dx}$ if $x^2 - 2xy + y^2 - 2x = 0$

Soln: $x^2 - 2xy + y^2 - 2x = 0$

$\Rightarrow \frac{d}{dx}(x^2) - 2\frac{d}{dx}(xy) + \frac{d}{dx}(y^2) - 2\frac{d}{dx}(x) = \frac{d}{dx}(0)$

$\Rightarrow 2x - 2\left(x \frac{dy}{dx} + y \cdot \frac{d}{dx} x\right) + 2y \cdot \frac{dy}{dx} - 2 = 0$

$\Rightarrow 2x - 2\left(x \frac{dy}{dx} + y \cdot 1\right) + 2y \frac{dy}{dx} - 2 = 0$

$\Rightarrow 2x - 2x \frac{dy}{dx} - 2y + 2y \frac{dy}{dx} - 2 = 0$

$\Rightarrow x - x \frac{dy}{dx} - y + y \frac{dy}{dx} - 1 = 0$ [Dividing by 2 on both sides]

$\Rightarrow \frac{dy}{dx}(y - x) = y - x + 1$

$\Rightarrow \frac{dy}{dx} = \frac{y - x + 1}{y - x}$ //

Note: When the relation between x and y are given by the equation of the form $f(x,y) = 0$, we are to find derivative of both sides of the equation w.r.t x and then $\frac{dy}{dx}$ is determined