Unit 2 - Calculus (I)

Mathematics function: Mathematics function is an expression which defines a relationship between an independent variable with another dependent variable. It is denoted by $y = f(x)$; where x is independent variable and y is dependent variable.

§ Types of Mathematics function:

Linear function: A function of the form $y = f(x) = ax + b$; $(a \neq 0)$ where x is independent variable and y is dependent variable, is called linear function. Linear function are those whose graph is a straight line.

eg. $y = 2x + 3$

Quadratic function: A function of the form $y = ax^2 + bx + c$; $(a \neq 0)$ where x is independent variable and y is dependent variable, is called quadratic function. In a quadratic function the highest power the variable is 2(two). The graph of a quadratic function is a parabola. If a is positive (>0), the graph opens upward and if a is negative $(a<0)$ then it opens downwards.

eg. $y = f(x) = 3x^2 - 5x + 3$

Polynomial function: A polynomial function is a function such, a quadratic function, a cubic function, a quartic function and so on involving only non-negative integer powers of x.

A polynomial of degree n is a function of the form -

$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$

where the a's (the coefficient of the polynomial) are real numbers.

(2)

Exponential function: A relation of the form $y = f(x) = a^x$; $(a>0, a\neq 1)$ where independent variable x ranging over the entire real no. line as the exponent of a positive no. a, is called exponential function. Probably the most important of the exponential function is $y = e^x$. It also sometimes written as $y = \exp(x)$ and it also defined as the sum of the infinite series, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

and $\quad e^{-x} = 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!} - \cdots$

Logarithmic function: Logarithmic functions are the inverse of the exponential function $y = a^x$ ie $x = a^y$. The logarithmic function $y = \log_a x$ is defined to be equivalent to the exponential function $x = a^y$

ie if $x = a^y \Rightarrow y = \log_a x$

Example: If $f(x) = a.\frac{x-b}{a-b} + b.\frac{x-a}{b-a}$; $(a \neq b)$ Then find $f(a)$, $f(b)$ and $f(a+b)$.

Solution: Given,

$$f(x) = a.\frac{x-b}{a-b} + b.\frac{x-a}{b-a}$$

$$\therefore f(a) = a.\frac{\cancel{(a-b)}}{\cancel{(a-b)}} + b.\frac{(a-a)}{(b-a)}$$

$$= a.1 + b.\frac{0}{(b-a)}$$

$$= a + 0$$

$$= a$$

$$f(b) = a.\frac{(b-b)}{(a-b)} + b.\frac{\cancel{(b-a)}}{\cancel{(b-a)}}$$

$$= a.\frac{0}{(a-b)} + b.1$$

$$= 0 + b$$

$$= b$$

$$f(a+b) = a\cdot\frac{a+\cancel{b}-\cancel{b}}{a-b} + b\cdot\frac{\cancel{a}+b-\cancel{a}}{b-a}$$

$$= a\cdot\frac{a}{a-b} + b\cdot\frac{b}{b-a}$$

$$= \frac{a^2}{a-b} + \frac{b^2}{b-a}$$

$$= \frac{a^2}{a-b} - \frac{b^2}{a-b} \quad [\because b-a = -(a-b)]$$

$$= \frac{a^2-b^2}{a-b} \quad [\because a^2-b^2 = (a+b)(a-b)]$$

$$= \frac{(a+b)\cancel{(a-b)}}{\cancel{(a-b)}}$$

$$= a+b$$

<u>Example</u>: If $f(x) = 2x^4+3x^2+1$, then show that $f(-x) = f(x)$

<u>Solution</u>: Given $f(x) = 2x^4+3x^2+1$

$$\therefore f(-x) = 2(-x)^4 + 3\cdot(-x)^2+1$$

$$= 2\cdot x^4 + 3\cdot x^2 + 1$$

$$= f(x)$$

$$\therefore f(-x) = f(x)$$

<u>Example</u>: If $f(x) = \frac{1-x}{1+x}$, Then find $\frac{f(x+h)-f(x)}{h}$

<u>Solution</u>: Given $f(x) = \frac{1-x}{1+x}$

$$\therefore f(x+h) = \frac{1-(x+h)}{1+(x+h)}$$

$$= \frac{1-x-h}{1+x+h}$$

Now, $\frac{f(x+h)-f(x)}{h}$

$$= \frac{\frac{1-x-h}{1+x+h} - \frac{1-x}{1+x}}{h}$$

$$= \frac{\frac{(1+x)(1-x-h) - (1-x)(1+x+h)}{(1+x+h)(1+x)}}{h}$$

$$= \frac{[1\cdot(1-x-h) + x(1-x-h)] - [1\cdot(1+x+h) - x\cdot(1+x+h)]}{h(1+x)(1+x+h)}$$

$$= \frac{(1-x-h+x-x^2-hx) - (1+x+h-x-x^2-hx)}{h(1+x)(1+x+h)}$$

$$\Rightarrow \frac{f(x+h)-f(x)}{h} = \frac{\cancel{1}-\cancel{x}-h+\cancel{x}-\cancel{x^2}-\cancel{hx}-\cancel{1}-\cancel{x}-h+\cancel{x}+\cancel{x^2}+\cancel{hx}}{h(1+x)(1+x+h)}$$

$$= \frac{-2h}{h(1+x)(1+x+h)}$$

$$= -\frac{2}{(1+x)(1+x+h)}$$

Example: If $f(x) = 2x^2-5x+4$, Then for what value of x $2f(x) = f(2x)$

Solution: Given, $f(x) = 2x^2-5x+4$

$\therefore 2.f(x) = 2.(2x^2-5x+4)$
$= 4x^2-10x+8$

~~Again~~, $f(2x) = 2.(2x)^2-5.(2x)+4$
$= 2.4x^2-10x+4$
$= 8x^2-10x+4$

A/Q, $2f(x) = f(2x)$
$\Rightarrow 4x^2-\cancel{10x}+8 = 8x^2-\cancel{10x}+4$
$\Rightarrow 8x^2-4x^2 = 8-4$
$\Rightarrow 4x^2 = 4$
$\Rightarrow x^2 = 1$
$\Rightarrow x = \pm 1$

Example: Show that The function $f(x) = \frac{x^2-x-2}{x^2+4x-12}$ is not defined for $x=2$

Solution: Given $f(x) = \frac{x^2-x-2}{x^2+4x-12}$

$\therefore f(2) = \frac{2^2-2-2}{2^2+4.2-12}$

$= \frac{4-4}{4+8-12}$

$= \frac{0}{0}$, which is not defined

$\therefore f(x) = \frac{x^2-x-2}{x^2+4x-12}$ is not defined for $x=2$

Example: If $f(x) = e^{mx}$; m is constant, Then show that $f(a).f(b).f(c) = f(a+b+c)$

Solution: Given, $f(x) = e^{mx}$

$\therefore \quad f(a) = e^{ma}, f(b) = e^{mb}, f(c) = e^{mc}$

$$
\begin{aligned}
LHS &= f(a).f(b).f(c) \\
&= e^{ma}.e^{mb}.e^{mc} \\
&= e^{ma+mb+mc} \\
&= e^{m(a+b+c)} \\
&= f(a+b+c) = RHS
\end{aligned}
$$

Example: If $f(x) = e^{px+q}$, (p, q are constant), Then show that $f(a).f(b).f(c) = f(a+b+c).e^{2q}$

Solution: Given, $f(x) = e^{px+q}$

$\therefore f(a) = e^{pa+q}, f(b) = e^{pb+q}, f(c) = e^{pc+q}$

$$
\begin{aligned}
\therefore LHS &= f(a).f(b).f(c) \\
&= e^{pa+q}.e^{pb+q}.e^{pc+q} \\
&= e^{(pa+q+pb+q+pc+q)} \\
&= e^{\{p(a+b+c)+q\}+2q} \\
&= e^{p(a+b+c)+q}.e^{2q} \\
&= f(a+b+c).e^{2q} = RHS
\end{aligned}
$$

Example: If $f(x) = \log x$, Prove that i) $f(abc) = f(a) + f(b) + f(c)$

ii) $f\left(\frac{a}{b}\right) = f(a) - f(b)$

Solution: Given, $f(x) = \log x$

i) $LHS = f(abc) = \log(abc) = \log a + \log b + \log c = f(a) + f(b) + f(c) = RHS$

ii) $f\left(\frac{a}{b}\right) = \log\left(\frac{a}{b}\right) = \log a - \log b = f(a) - f(b) = RHS$