Unconstrained Maxima and Minima

B.A 2ND Sem (Honours)

The term optimization is most frequently in economic analysis. The basic economic problems centre around the problem of choice and we have to choose between the alternatives in order to optimize certain objectives. For example, it may be a problem of determining price quantity combination to maximize profit of a firm. We can have a problem of maximizing total utility from a basket of consumption goods subject to limited income of the consumer or it may be a problem of choosing optimum combination of inputs in order to minimize the cost of production subject to a given production function.

RELATIVE AND ABSOLUTE EXTREMA

In mathematical language, a common term used to indicate the maximum and minimum values is ‘extrema’ meaning simply an extreme value. When we plot a function on graph paper, we are likely to have either of the two extreme values – maximum and minimum –or both of them. We can have many such extreme values throughout the graph of function as shown in figure-(8.1).

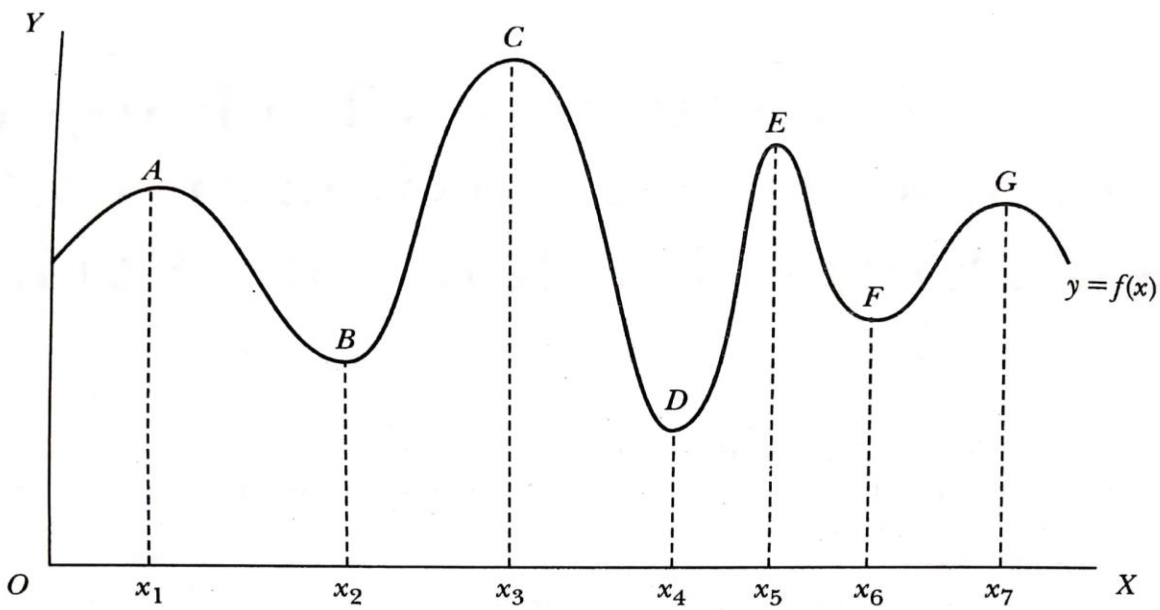

Figure 8.1

But in contrast, the points *A*, *E* and *G* represent relative maximum because they represent extreme value (maximum) in the immediate neighbourhood of *A*, *E* and *G*. In order words, the relative maximum value of the function $y = f(x)$ in a particular segment of the curve. Similarly, *B* and *F* represent the relative minimum of the function $y = f(x)$ as they represent minimum value of the function in the neighbourhood of *B* and *F*. So we can have only one absolute maximum or/and absolute minimum, but we can have more than one relative maximum or/and relative minimum of the function as shown in the figure (8.1).

In a function $y = g(x)$, if we have only one extreme point *P* in the whole range of the curve as shown in figure (8.2), *P* represents absolute maximum. The point *P* also represents the relative maximum of the function $y = g(x)$. This will also be true in case of a function where we have only one minimum value of the function representing both absolute and relative minimum.

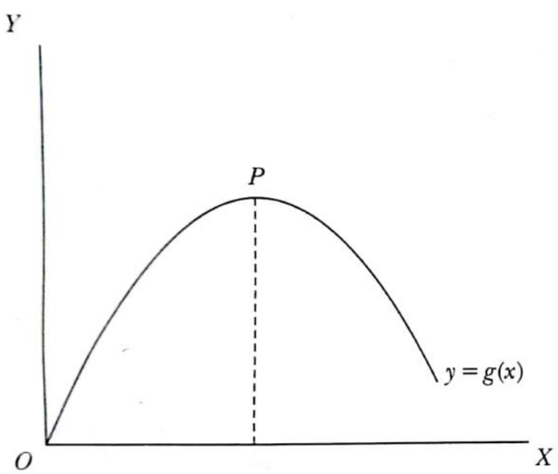

Figure 8.2

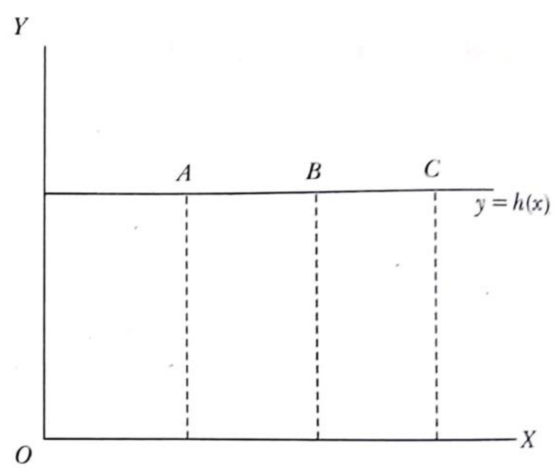

Figure 8.3

Sometimes we may have a function that, when plotted on graph paper, gives a horizontal line as shown in figure (8.3). All the values of *x* in such a function will give the same value of *y*. In such case all the points of the curve can be considered as maximum value or minimum value of the function-or indeed neither. So in case of such function, the concept of absolute and relative extrema is meaningless.

First order conditions for a maximum:

Consider the total revenue function:

TR = $60q - 0.2q2$

This will take an inverted U-shape similar shown in figure 9.1. If we ask the question 'when is TR at its maximum?' the answer is obviously at M, which is the highest point on the curve. At this maximum position the TR schedule is flat. To the left of M, TR is rising and has a positive slope, and to the right of M, the TR schedule is falling and has a negative slope. At M itself the slope is zero. We can therefore say that for a function *of this shape* the maximum point will be where its slope is zero. This zero slope requirement is a necessary ***first-order condition for a maximum.***

Zero slope will not guarantee that a function is at a maximum, as explained in the next section where the necessary additional second-order conditions are explained. However, in this particular example we know for certain that zero slope corresponds to the maximum value of the function.

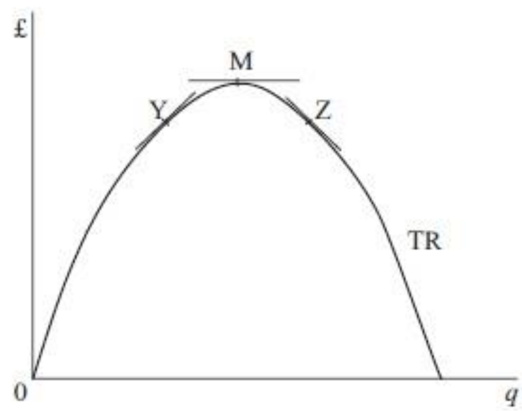

Figure 9.1

The slope is zero when

$60 - 0.4q = 0$

$60 = 0.4q$

$150 = q$

Therefore TR is maximized when quantity is 150.

Second-order condition for a maximum:

In the example given above, it was obvious that the TR function was a maximum when its slope was zero because we knew the function had an inverted U-shape. However, consider the function in figure 9.2(a). This has a slope of zero at N, but this is its minimum point not its maximum. In the case of the function in figure 9.2(b) the slope is zero at I, but this is neither a maximum nor a minimum point.

The examples in figure 9.2 clearly illustrate that although a zero slope is **necessary** for a function to be at its maximum it is not a **sufficient** condition. A zero slope just means that the function is at what is known as a 'stationary point', i.e. its slope is neither increasing nor decreasing. Some stationary points will be turning points, i.e. the slope changes from positive to negative (or vice versa) at these points, and will be maximum (or minimum) points of the function.

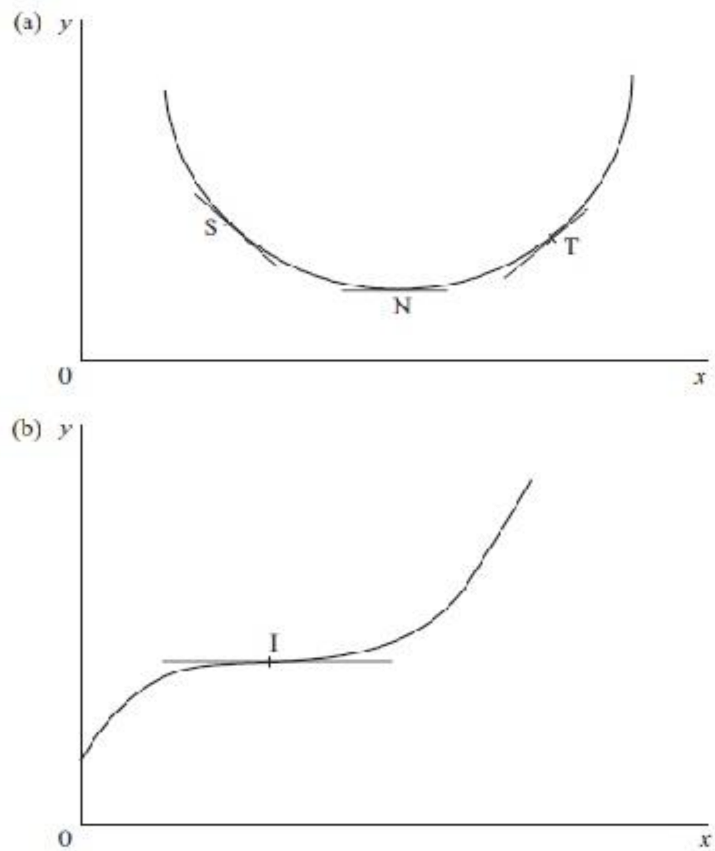

Figure 9.2

In order to find out whether a function is at a maximum or a minimum or a point of inflexion (as in figure 9.2(b) when its slope is zero we have to consider what are known as the *second order* conditions. (The first-order condition for any of the three forms of stationary point is that the slope of the function is zero.)

The second-order conditions tell us what is happening to the rate of change of the slope of the function. If the rate of change of the slope is negative it means that the slope decreases as the variable on the horizontal axis is increased. If the slope is decreasing and one is at a point where the actual slope is zero this means that the slope of the function is positive slightly to the left and negative slightly to the right of this point. This is the case in figure 9.1. The slope is positive at Y, zero at M and negative at Z. Thus, if the rate of change of the slope of a function is negative at the point where the actual slope is zero then that point is a maximum.

This is the second-order condition for a maximum. Until now, we have just assumed that a function is maximized when its slope is zero if a sketch graph suggests that it takes an inverted U-shape. From now on we shall make this more rigorous check of the second-order conditions to confirm whether a function is maximized at any stationary point.

It is a straightforward exercise to find the rate of change of the slope of a function. We know that the slope of a function $y = f(x)$ can be found by differentiation. Therefore if we differentiate the function for the slope of the original function, i.e. dy/dx, we get the rate of change of the slope. This is known as the **second-order derivative** and is written $d2y/dx2$.

Example 8.1 Find out relative extrema of the function $y = 20x - 2x^2$.

Solution: To find out the extreme value of the function $y = 20x - 2x^2$, we find the first order condition $f'(x) = 0$.

$$f'(x) = \frac{dy}{dx} = 20 - 4x = 0$$

$$\therefore \quad x = 5.$$

Now we are to verify whether the function will attain maximum or minimum when $x = 5$.

So we test the second order condition whether $f''(x) > 0$ or $f''(x) < 0$ for $x = 5$.

Now,

$$f''(x) = \frac{d^2y}{dx^2} = -4 < 0$$

$\therefore$ $x = 5$ will give maximum value of the function $y = 20x - 2x^2$.

So the maximum value of the function will be $20 \times 5 - 2 \times 5^2 = 100 - 50 = 50$.

Figure (8.6) below displays the diagrammatic representation of the relative maximum value of the said function. While plotted, the function gives a parabolic curve having the peak at A.

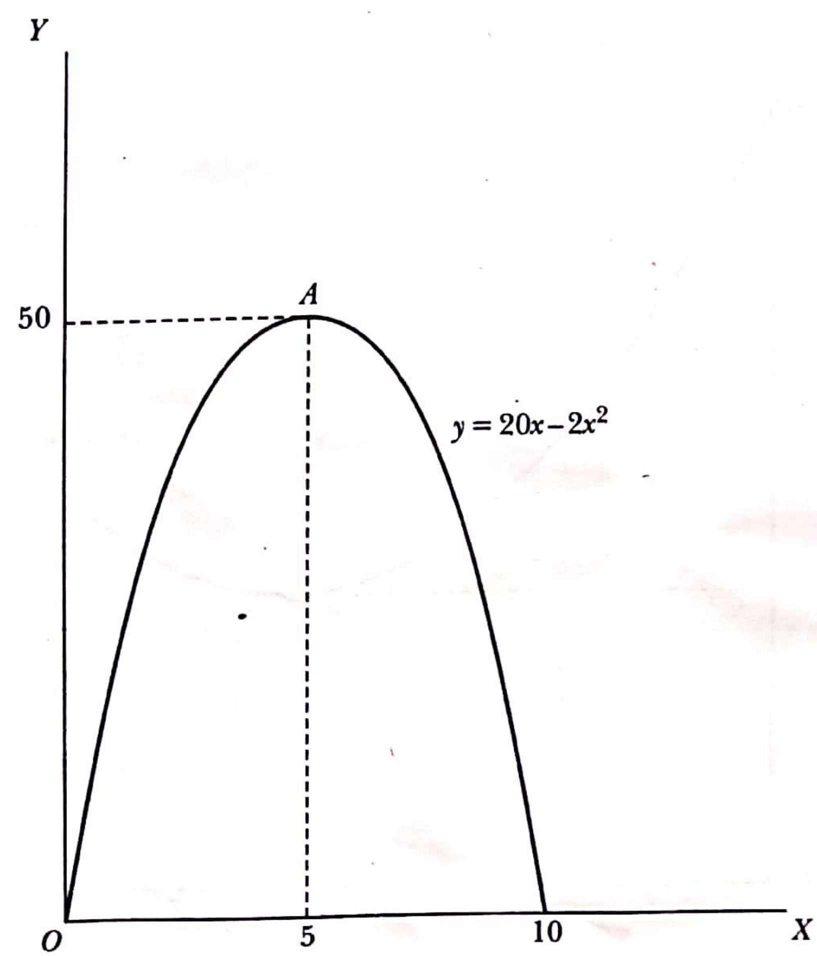

Figure 8.6

Example 8.2 Find the relative extrema of the function $y = 2x^2 - 16x + 50$.

Solution: In order to find out the extreme value of the function $y = (2x^2 - 16x + 50)$, we are required to satisfy the first order condition such that $\frac{dy}{dx} = 0$.

Thus $$\frac{dy}{dx} = 4x - 16 + 0 = 0, \text{ or } x = \frac{16}{4} = 4.$$

Now, to verify whether the function will attain maximum or minimum when $x = 4$ we find the second order derivative $\frac{d^2y}{dx^2}$ and see whether

$$\frac{d^2y}{dx^2} > 0 \quad \text{or} \quad \frac{d^2y}{dx^2} < 0.$$

Now $$\frac{d^2y}{dx^2} = 4 > 0.$$

Therefore, the function $y = 2x^2 - 16x + 50$ will have relative minimum when $x = 4$ and this is shown in figure (8.7) below.

So the minimum value of the function at $\bar{x} = 4$ is $y = 2(4)^2 - 16(4) + 50 = 18$.

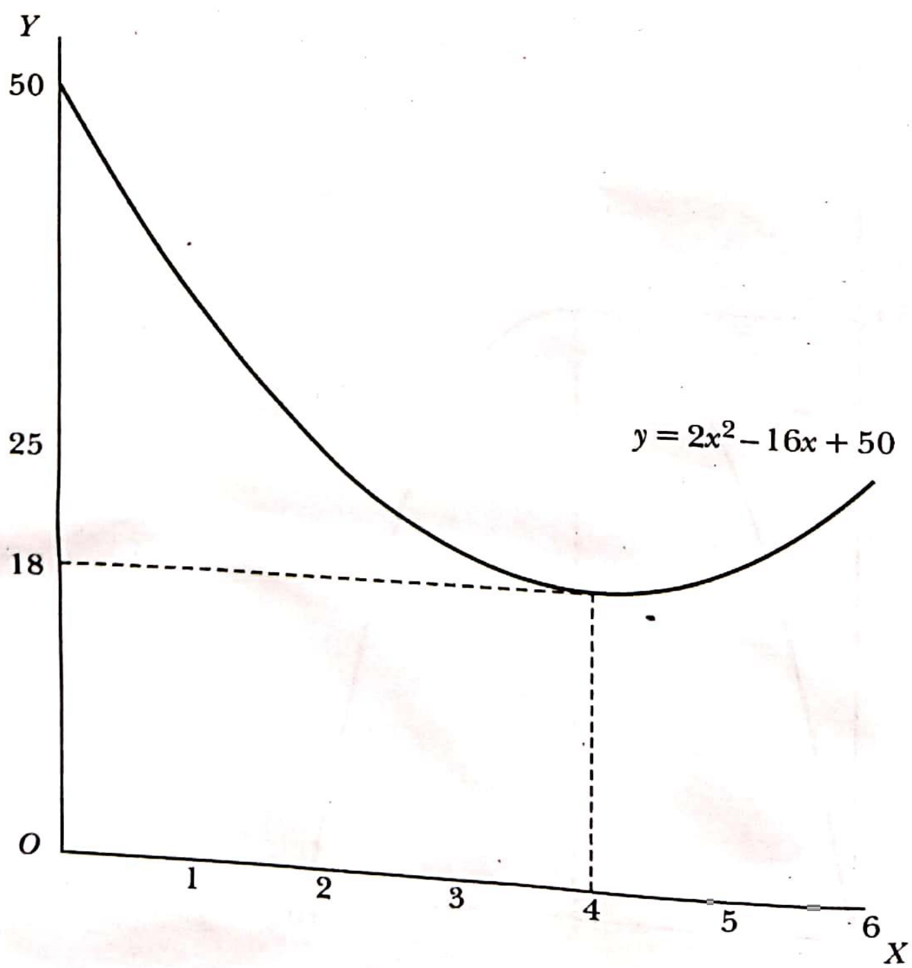

Figure 8.7

Example 8.3 Find the extreme values of the following function

$$y = x^3 - 9x^2 + 15x + 20.$$

Solution: The determination of extrema values requires that $\frac{dy}{dx} = f'(x) = 0.$

In our given function $y = x^3 - 9x^2 + 15x + 20$

$\frac{dy}{dx} = 3x^2 - 18x + 15 = 0$ (since $f'(x) = 0$). The value of x is given by the solution of the quadratic equation

$$3x^2 - 18x + 15 = 0$$

$$\therefore \quad x = \frac{18 \pm \sqrt{(18)^2 - 4(3)(15)}}{2(3)}$$

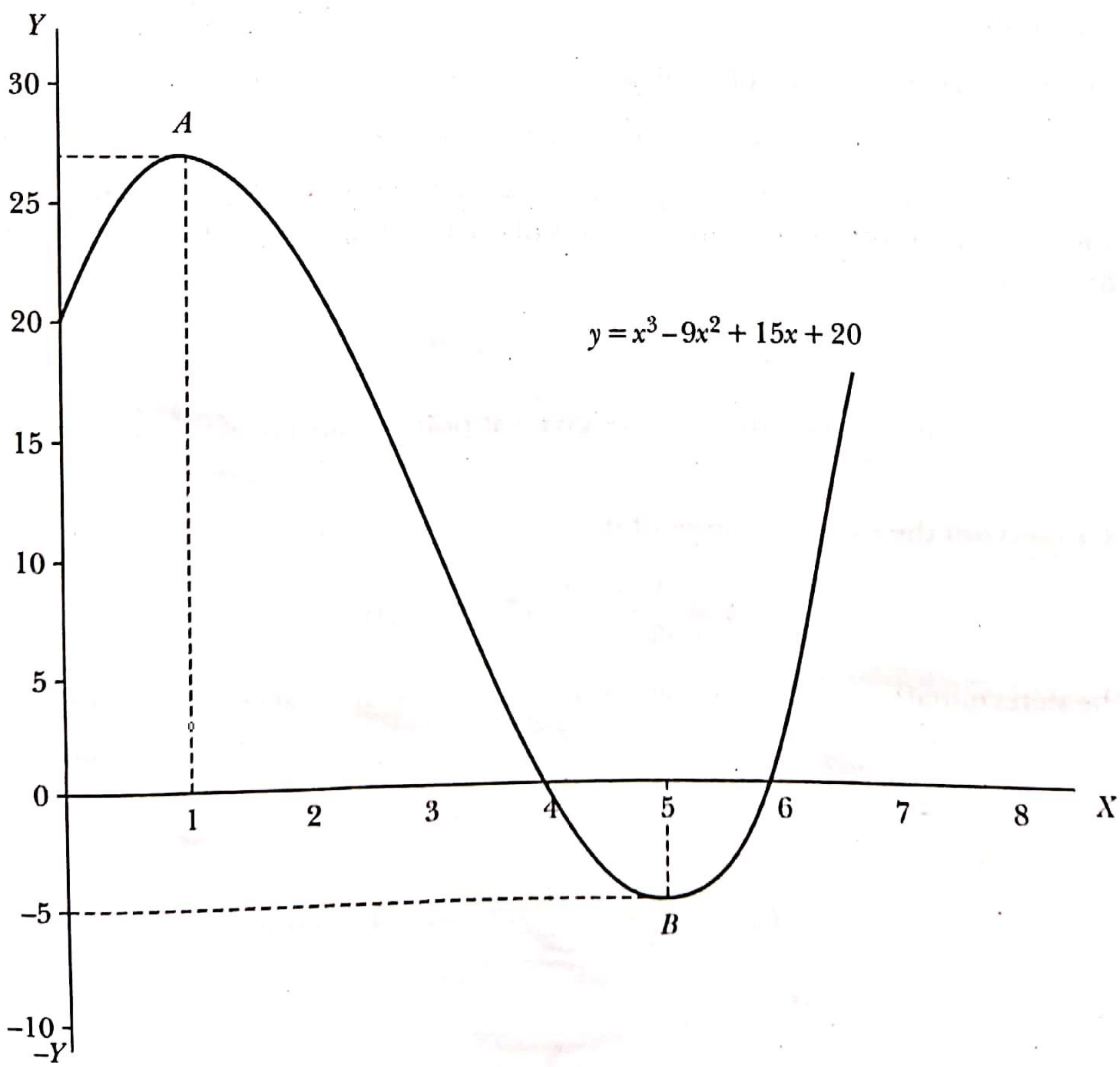

Figure 8.8

$$= \frac{18 \pm \sqrt{144}}{6} = \frac{18 \pm 12}{6}$$

$$= \frac{6}{6} \text{ or } \frac{30}{6}$$

$$= 1 \text{ or } 5$$

One of the values of x will give maximum value of y and the other will give the minimum value of y. In order to ascertain the value of x maximizing/minimizing y, we find out the second order derivative of y with respect to x and see whether $\frac{d^2y}{dx^2}$ is greater than zero or less than zero for a particular value of x.

Now $$f''(x) = \frac{d^2y}{dx^2} = 6x - 18.$$

If $x = 1, f''(x) = -12 < 0$ and x will maximize the function. If $x = 5, f''(x) = 12 > 0$ and so x will minimize the function.

The maximum (relative) value of y will be

$$y = (1)^3 - 9(1)^2 + 15(1) + 20 = 27.$$

Again relative minimum value of y is given by $y = (5)^3 - 9(5)^2 + 15(5) + 20 = -5$. The graphic representation of maximum and minimum points of the function $y = x^3 - 9x^2 + 15x + 20$ are shown in figure (8.8) below.

The relative maximum and minimum are given at points A and B of the curve when $x = 1$ and $x = 5$ respectively.

Example 8.4 Find out the extreme values of the following function

$$y = \frac{1}{3}x^3 - \frac{1}{2}5x^2 + 4x + 10.$$

Solution: The determination of extreme values need to satisfy the first order condition $f'(x) = 0$.

In our given function

$$y = \frac{1}{3}x^3 - \frac{1}{2}5x^2 + 4x + 10,$$

$$f'(x) = \frac{1}{3}3x^2 - \frac{1}{2} \times 2 \times 5x + 4 + 0 = 0$$

or $$x^2 - 5x + 4 = 0$$

$\therefore$ $$x = \frac{5 \pm \sqrt{(5)^2 - 4(4)}}{2 \times 1}$$

$$= \frac{5 \pm \sqrt{9}}{2} = \frac{5 \pm 3}{2}$$

$$= 4 \quad \text{or} \quad 1.$$

Now $f''(x) = 2x - 5$. When $x = 4, f''(x) = 8 - 5 = 3 > 0$, the function will attain minimum value; when $x = 1, f''(x) = 2 - 5 = -3 < 0$, the function will attain maximum value.

So the maximum value of y is derived by substituting $x = 1$ and minimum value by substituting $x = 4$ in the function $y = \frac{1}{3}x^3 - \frac{1}{2}5x^2 + 4x + 10.$

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